

# SUSY QM, symmetries and spectrum generating algebras for two-dimensional systems

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## Abstract

We show in a systematic and clear way how factorization methods can be used to construct the generators for hidden and dynamical symmetries. This is shown by studying the 2D problems of hydrogen atom, the isotropic harmonic oscillator and the radial potential  $A\rho^{2\zeta-2} - B\rho^{\zeta-2}$ . We show that in these cases the non-compact (compact) algebra corresponds to  $so(2, 1)$  ( $su(2)$ ).

*Keywords* : SUSY QM; Spectrum generating algebras; Symmetry; Lie algebras.

## 1 Introduction

Since their introduction the factorization methods have played an important roll in the study of quantum systems [1, 2, 3]. This is because, if the Schrödinger equation is factorizable, the energy spectrum and the eigenfunctions are obtained algebraically. Infeld and Hull [4], generalized the

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ideas of Dirac [1] and Schrödinger [2, 3] and created a new factorization method (IHFM) which uses a particular solution for the Ricatti equation. This method allowed to classify the problems according to the characteristics involved in the potentials. A variant of the IHFM, proposed by Mielnik [5], gives the possibility to obtain isospectral Hamiltonians. On the other hand, the charges in  $N = 2$  SUSY QM are written in terms of the operators which factorize the partner Hamiltonians  $H_+$  and  $H_-$  [6, 7, 8].

For the most important central potential problems (the hydrogen atom and the isotropic harmonic oscillator) algebraic treatments by means of non-compact and compact groups are known [9, 10], being the last ones generally related to hidden symmetries and degeneracy [11]-[17]. However, a systematic method to find both compact and non-compact group generators for a given system has not still developed. These generators have been intuitively found and forced them to close an algebra, as it is extensively shown in reference [18].

The link between SUSY QM and hidden symmetries has been studied by showing that the factorization operators of the partner Hamiltonians are contained within the symmetry group operators [19]-[25]. This connection has been achieved for the two- and three-dimensional hydrogen atom [19, 20], the two- and three-dimensional isotropic harmonic oscillator [21, 22] and for a neutron in the magnetic field of a linear current [23]-[25]. It must be emphasized that the charges are written in terms of the superpotential which is involved in the factorization operators of the Hamiltonian  $H_+$ . Thus, the symmetries of the Hamiltonian  $H_+$  play an important role in the construction of operators which define a superalgebra of the SUSY Hamiltonian  $H = \text{diag}(H_+, H_-)$ . Also, hidden supersymmetry, related with additional supersymmetries specific to the dynamics, has been associated with the existence of a supergroup ( a dynamical superalgebra ) [26]-[31].

The purpose of this paper is to show that the IHFM allows us to construct the generators of symmetries (compact algebras), whereas the Schrödinger factorization allows us to find dynamical symmetries (spectrum generating algebras).

This work is organized as follows. In Sections 2 and 3 we study, from the factorization methods approach, the two-dimensional isotropic harmonic oscillator (TDIHO) and the two-dimensional hydrogen atom (TDHA), respectively. By means of the Schrödinger factorization we obtain in a systematic way the corresponding non-compact algebra generators and show that these systems have an  $so(2,1)$  symmetry. On the other hand, for each problem

we study how the constants of motion and the radial SUSY operators are related. In section 4 we use the results from the previous sections to find the dynamical and hidden symmetries for the radial potential  $A\rho^{2\zeta-2} - B\rho^{\zeta-2}$ , which generalizes the former ones [32, 33, 34]. Finally, in Section 5 we give the concluding remarks.

## 2 Two-dimensional isotropic harmonic oscillator

### 2.1 Schrödinger factorization

The Schrödinger equation for the two-dimensional isotropic harmonic oscillator (TDIHO) in polar coordinates  $(\rho, \varphi)$  is

$$H_{h.o.}\psi(\rho, \varphi) \equiv \left[ -\frac{\hbar^2}{2\mu} \left( \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2} \right) + \frac{\hbar\omega}{2} \beta^2 \rho^2 \right] \psi(\rho, \varphi) = E\psi(\rho, \varphi), \quad (1)$$

where  $\beta = \sqrt{\mu\omega/\hbar}$ . In what follows, we will set  $\hbar = \omega = 1$  and assume that the mass involved in each system under study is  $\mu = 1$ .

Considering the conservation of the operator  $L_z$ , the eigenfunction can be written as

$$\psi_{nm}(\rho, \varphi) = R_{nm}(\rho)e^{im\varphi} \equiv \rho^{-\frac{1}{2}}u_{nm}(\rho)e^{im\varphi}, \quad (2)$$

where  $m = 0, \pm 1, \pm 2, \dots$  is the quantum number corresponding to the operator  $L_z$ . Therefore, from equation (1) we obtain

$$H_m u_{nm} \equiv -\frac{1}{2} \frac{d^2 u}{d\rho^2} + \left[ \frac{1}{2} \rho^2 + \frac{1}{2} \frac{m^2 - \frac{1}{4}}{\rho^2} \right] u_{nm} = E_n u_{nm}. \quad (3)$$

This equation can be written as

$$\left( \rho^2 \frac{d^2}{d\rho^2} - \rho^4 + 2E_n \rho^2 \right) u_{nm} = \left( m^2 - \frac{1}{4} \right) u_{nm}. \quad (4)$$

We propose a pair of first order operators such that

$$\left( \rho \frac{d}{d\rho} + a\rho^2 + b \right) \left( \rho \frac{d}{d\rho} + c\rho^2 + f \right) u_{nm} = g u_{nm}, \quad (5)$$

where  $a$ ,  $b$ ,  $c$ ,  $f$  and  $g$  are constants to be determined. By expanding this expression and comparing it with Eq. (4), we obtain

$$a = -c = \mp 1, \quad b = -f = -1 \pm E_n, \quad g = m^2 - f^2. \quad (6)$$

Using these results, Eq. (5) can be expressed in the following way

$$(D_-^n - 1)D_+^n u_{nm} = \frac{1}{4} \left[ \left( E_n + \frac{1}{2} \right) \left( E_n + \frac{3}{2} \right) - \left( m^2 - \frac{1}{4} \right) \right] u_{nm}, \quad (7)$$

$$(D_+^n + 1)D_-^n u_{nm} = \frac{1}{4} \left[ \left( E_n - \frac{1}{2} \right) \left( E_n - \frac{3}{2} \right) - \left( m^2 - \frac{1}{4} \right) \right] u_{nm}, \quad (8)$$

where we have defined

$$D_\pm^n = \frac{1}{2} \left( \mp \rho \frac{d}{d\rho} + \rho^2 - E_n \mp \frac{1}{2} \right). \quad (9)$$

If we consider that the energy spectrum of the TDIHO is  $E_n = n + 1$  then, from (9) the following recursion relations hold

$$D_+^{n\pm 2} = D_+^n \mp 1, \quad (10)$$

$$D_-^{n\pm 2} = D_-^n \mp 1. \quad (11)$$

By performing the change  $n \rightarrow n - 2$  in Eq. (7) and  $n \rightarrow n + 2$  in Eq. (8), we have

$$\begin{aligned} (D_-^{n-2} - 1)D_+^{n-2} u_{n-2\,m} &= D_-^n (D_+^n + 1) u_{n-2\,m} \\ &= \frac{1}{4} \left[ \left( E_n - \frac{1}{2} \right) \left( E_n - \frac{3}{2} \right) - \left( m^2 - \frac{1}{4} \right) \right] u_{n-2\,m}, \end{aligned} \quad (12)$$

and

$$\begin{aligned} (D_+^{n+2} + 1)D_-^{n+2} u_{n+2\,m} &= D_+^n (D_-^n - 1) u_{n+2\,m} \\ &= \frac{1}{4} \left[ \left( E_n + \frac{1}{2} \right) \left( E_n + \frac{3}{2} \right) - \left( m^2 - \frac{1}{4} \right) \right] u_{n+2\,m}, \end{aligned} \quad (13)$$

respectively. Multiplying (7) by  $D_+^n$  and (8) by  $D_-^n$ , and comparing the resulting expressions with (12) and (13), it is shown that

$$D_+^n u_{nm} \propto u_{n+2\,m}, \quad (14)$$

$$D_-^n u_{nm} \propto u_{n-2\,m}. \quad (15)$$

The action of these operators on the radial function  $u_{nm}$  is shown in figure 1. To find the dynamical symmetry algebra, we define the operator

$$D_3 = \frac{1}{4} \left( -\frac{d^2}{d\rho^2} + \rho^2 + \frac{m^2 - \frac{1}{4}}{\rho^2} \right), \quad (16)$$

which satisfies

$$D_3 u_{nm} = \frac{E_n}{2} u_{nm}. \quad (17)$$

Thus, it allows us to define the new pair of operators

$$D_{\pm} = \frac{1}{2} \left( \mp \rho \frac{d}{d\rho} + \rho^2 - 2D_3 \mp \frac{1}{2} \right), \quad (18)$$

which are independent of  $E_n$ .

We show that the operators  $D_{\pm}$  and  $D_3$  satisfy the  $so(2, 1)$  Lie algebra

$$[D_{\pm}, D_3] = \mp D_{\pm}, \quad (19)$$

$$[D_+, D_-] = -2D_3. \quad (20)$$

Thus, from the Schrödinger factorization we have obtained the generators of the non-compact algebra for the TDIHO. In Section 3, a similar procedure will be applied to find the spectrum generating algebra for the TDHA.

## 2.2 SUSY QM

Applying SUSY QM to the Hamiltonian  $H_m$  defined in Eq. (3), we find that the SUSY operators are

$$B_{1,2} = \frac{1}{\sqrt{2}} \left( \frac{d}{d\rho} \pm \frac{m \pm \frac{1}{2}}{\rho} - \rho \right), \quad (21)$$

$$B_{1,2}^{\dagger} = \frac{1}{\sqrt{2}} \left( -\frac{d}{d\rho} \pm \frac{m \pm \frac{1}{2}}{\rho} - \rho \right). \quad (22)$$

From now on, the indices 1 and 2 will correspond to the upper and the lower sign in the expression where they appear. Therefore, we construct the partner Hamiltonians

$$H_{1,2+} \equiv B_{1,2} B_{1,2}^{\dagger} = H_m \mp (m \pm 1), \quad (23)$$

$$H_{1,2-} \equiv B_{1,2}^{\dagger} B_{1,2} = H_{m \pm 1} \mp m. \quad (24)$$

By using Eqs. (23) and (24), we show that

$$B_1^\dagger u_{n\ m+1} \propto u_{n+1\ m}, \quad (25)$$

$$B_1 u_{nm} \propto u_{n-1\ m+1} \quad (26)$$

and

$$B_2^\dagger u_{n\ m-1} \propto u_{n+1\ m}, \quad (27)$$

$$B_2 u_{nm} \propto u_{n-1\ m-1}. \quad (28)$$

Thus, the actions of the SUSY operators of the TDIHO on the eigenstates  $u_{nm}$  of equation (3) are to change the values of both quantum numbers simultaneously, and they are different among them. On the other hand, because these operators change the principal quantum number, they are not directly related to the constants of motion of the system, as it will be shown in the next section.

### 2.3 SUSY QM and constants of motion

It is well known that the operators

$$a_{1,2} = \frac{1}{2} e^{\pm i\varphi} \left( \frac{\partial}{\partial \rho} \pm \frac{i}{\rho} \frac{\partial}{\partial \varphi} + \rho \right), \quad (29)$$

$$a_{1,2}^\dagger = \frac{1}{2} e^{\mp i\varphi} \left( -\frac{\partial}{\partial \rho} \pm \frac{i}{\rho} \frac{\partial}{\partial \varphi} + \rho \right) \quad (30)$$

act on the complete eigenfunctions of the TDIHO as follows [35]

$$\begin{aligned} a_2 \psi_{nm} &= \sqrt{\frac{n+m}{2}} \psi_{n-1\ m-1}, & a_2^\dagger \psi_{nm} &= \sqrt{\frac{n+m+2}{2}} \psi_{n+1\ m+1}, \\ a_1 \psi_{nm} &= \sqrt{\frac{n-m}{2}} \psi_{n-1\ m+1}, & a_1^\dagger \psi_{nm} &= \sqrt{\frac{n-m+2}{2}} \psi_{n+1\ m-1}. \end{aligned}$$

On the other hand, the constants of motion for this system, expressed in polar coordinates, are [13]

$$O_\pm = \mp \frac{i}{2} e^{\pm 2i\varphi} \left[ (1 \pm L_z) \left( \frac{1}{\rho} \frac{\partial}{\partial \rho} \mp \frac{L_z}{\rho^2} \right) + H_m \right], \quad (31)$$

$$O_3 = \frac{L_z}{2} = -\frac{i}{2} \frac{\partial}{\partial \varphi} \quad (32)$$

and satisfy the  $su(2)$  Lie algebra

$$[O_{\pm}, O_3] = \mp O_{\pm}, \quad (33)$$

$$[O_+, O_-] = 2O_3. \quad (34)$$

From these commutation relations, we prove that

$$O_{\pm} \psi_{nm} \propto \psi_{n \ m \pm 2}. \quad (35)$$

Also, it is straightforward to show that

$$O_+ = -ia_2^{\dagger} a_1, \quad O_- = ia_1^{\dagger} a_2. \quad (36)$$

In order to find how the SUSY operators and constants of motion of the TDIHO are related, we apply the operators (29) and (30) on the states defined in (2). Thus, we obtain

$$a_{1,2} \psi_{nm} = \frac{1}{2} \rho^{-\frac{1}{2}} e^{i(m \pm 1)\varphi} \mathcal{O}_{1,2}(m) u_{nm}, \quad (37)$$

$$a_{1,2}^{\dagger} \psi_{nm} = \frac{1}{2} \rho^{-\frac{1}{2}} e^{i(m \mp 1)\varphi} \mathcal{O}_{1,2}^{\dagger}(m) u_{nm}, \quad (38)$$

where

$$\mathcal{O}_{1,2}(m) = \frac{d}{d\rho} \mp \frac{m \pm \frac{1}{2}}{\rho} + \rho, \quad (39)$$

$$\mathcal{O}_{1,2}^{\dagger}(m) = -\frac{d}{d\rho} \mp \frac{m \mp \frac{1}{2}}{\rho} + \rho. \quad (40)$$

From the expressions above, the following identities hold

$$\mathcal{O}_1(m) = -\sqrt{2} B_1^{\dagger}, \quad \mathcal{O}_1^{\dagger}(m+1) = -\sqrt{2} B_1, \quad (41)$$

$$\mathcal{O}_2(m) = -\sqrt{2} B_2^{\dagger}, \quad \mathcal{O}_2^{\dagger}(m-1) = -\sqrt{2} B_2. \quad (42)$$

This shows that the SUSY operators are directly contained in the operators  $a_{1,2}$  and  $a_{1,2}^{\dagger}$ . Therefore, from the expressions (36), the radial spherical constants of motion for the TDIHO are a product of radial SUSY operators.

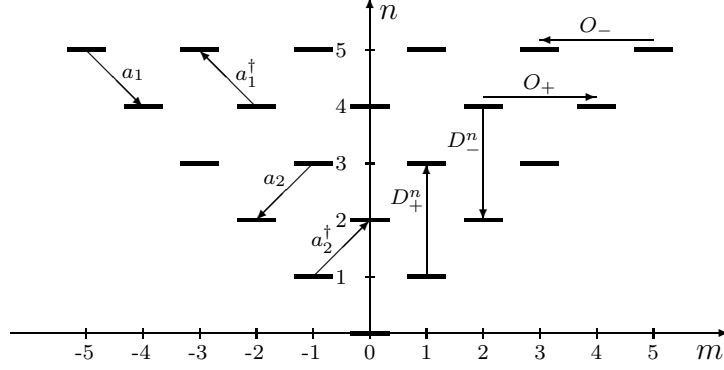


Figure 1. Action of the operators  $D_{\pm}^n$ ,  $O_{\pm}$ ,  $a_{1,2}$  and  $a_{1,2}^{\dagger}$  on the states of the TDIHO.

### 3 Two-dimensional hydrogen atom

#### 3.1 Schrödinger factorization

The Schrödinger equation for the two-dimensional hydrogen atom is

$$H_{h.a.}\tilde{\psi}(\rho, \varphi) \equiv \left[ -\frac{1}{2} \left( \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2} \right) - \frac{1}{\rho} \right] \tilde{\psi}(\rho, \varphi) = \tilde{E} \tilde{\psi}(\rho, \varphi). \quad (43)$$

where we have set the electric charge equal to unity.

Since  $[H_{h.a.}, L_z] = 0$ , the eigenfunctions are written as

$$\tilde{\psi}_{nm}(\rho, \varphi) = \tilde{R}_{nm}(\rho) e^{im\varphi} \equiv \rho^{-\frac{1}{2}} \tilde{u}_{nm}(\rho) e^{im\varphi} \quad (44)$$

and the radial Hamiltonian for this system satisfies

$$\tilde{H}_m \tilde{u}_{nm} \equiv -\frac{1}{2} \frac{d^2 u}{d\rho^2} + \left[ -\frac{1}{\rho} + \frac{1}{2} \frac{m^2 - \frac{1}{4}}{\rho^2} \right] \tilde{u}_{nm} = \tilde{E}_n \tilde{u}_{nm}. \quad (45)$$

For bound states, we define  $K_n^2 = -\frac{1}{2E_n}$  and  $\rho = K_n x$ . Thus, equation (45) can be rewritten as

$$\left( x^2 \frac{d^2}{dx^2} + 2K_n x - x^2 \right) \tilde{u}_{nm} = \left( m^2 - \frac{1}{4} \right) \tilde{u}_{nm}. \quad (46)$$



Following the procedure applied to the TDIHO, the Schrödinger operators that change the principal quantum number  $n$  are

$$T_{\pm}^n = \mp x \frac{d}{dx} + x - K_n, \quad (47)$$

which satisfy

$$(T_-^n - 1)T_+^n \tilde{u}_{nm} = \left[ K_n(K_n + 1) - \left( m^2 - \frac{1}{4} \right) \right] \tilde{u}_{nm}, \quad (48)$$

$$(T_+^n + 1)T_-^n \tilde{u}_{nm} = \left[ K_n(K_n - 1) - \left( m^2 - \frac{1}{4} \right) \right] \tilde{u}_{nm} \quad (49)$$

and their recurrence relations are given by

$$T_+^n \tilde{u}_{nm} \propto \tilde{u}_{n+1\ m}, \quad (50)$$

$$T_-^n \tilde{u}_{nm} \propto \tilde{u}_{n-1\ m}. \quad (51)$$

From (46), we define the operator

$$T_3 = \frac{1}{2} \left( -x \frac{d^2}{dx^2} + x + \frac{m^2 - \frac{1}{4}}{x} \right), \quad (52)$$

which allows us to generalize  $T_{\pm}^n$  to

$$T_{\pm} = \mp x \frac{d}{dx} + x - T_3. \quad (53)$$

We show that these operators close the  $so(2, 1)$  Lie algebra

$$[T_{\pm}, T_3] = \mp T_{\pm}, \quad (54)$$

$$[T_+, T_-] = -2T_3. \quad (55)$$

Notice that the non-compact algebras generators (Eqs. (16), (18), (52) and (53)) were introduced previously in Ref. [36] for the case of the three-dimensional problems. However, the origin of these operators was not investigated.

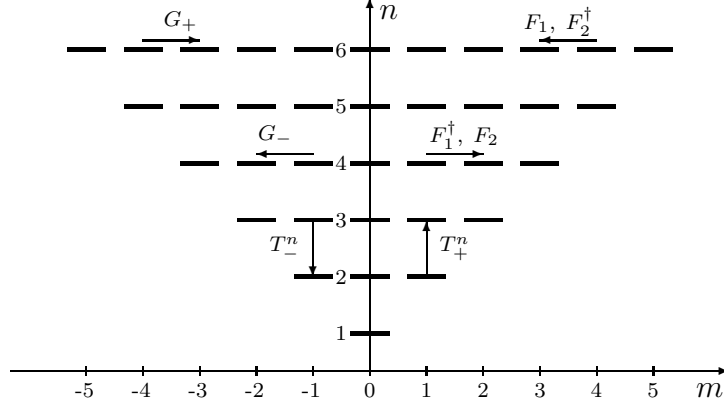


Fig 2. Action of the operators  $T_{\pm}^n$ ,  $G_{\pm}$ ,  $F_{1,2}$  and  $F_{1,2}^{\dagger}$  on the states of the TDHA.

### 3.2 SUSY QM

For the TDHA the SUSY operators which factorize the Hamiltonian  $\tilde{H}_m$  are

$$F_{1,2} = \frac{1}{\sqrt{2}} \left[ \frac{d}{d\rho} \pm \frac{m \pm \frac{1}{2}}{\rho} \mp \frac{1}{m \pm \frac{1}{2}} \right], \quad (56)$$

$$F_{1,2}^{\dagger} = \frac{1}{\sqrt{2}} \left[ -\frac{d}{d\rho} \pm \frac{m \pm \frac{1}{2}}{\rho} \mp \frac{1}{m \pm \frac{1}{2}} \right] \quad (57)$$

and the partners Hamiltonians are given by

$$H_{1,2+} \equiv F_{1,2} F_{1,2}^{\dagger} = \tilde{H}_m + \frac{1}{2(m \pm \frac{1}{2})^2},$$

$$H_{1,2-} \equiv F_{1,2}^{\dagger} F_{1,2} = \tilde{H}_{m\pm 1} + \frac{1}{2(m \pm \frac{1}{2})^2}.$$

It is straightforward to show that, if  $\tilde{u}_{nm}$  satisfies (45), then

$$F_1 \tilde{u}_{nm} \propto \tilde{u}_{n-1, m}, \quad (58)$$

$$F_1^{\dagger} \tilde{u}_{nm} \propto \tilde{u}_{n+1, m+1} \quad (59)$$

and

$$F_2 \tilde{u}_{nm} \propto \tilde{u}_{n, m+1}, \quad (60)$$

$$F_2^{\dagger} \tilde{u}_{nm} \propto \tilde{u}_{n-1, m-1}. \quad (61)$$

The action of these operators on the radial function  $\tilde{u}_{nm}$  is shown in figure 2. From expressions above, we conclude that the action of the operators  $F_1$  and  $F_1^\dagger$  on  $\tilde{u}_{nm}$  is identical to that of the operators  $F_2^\dagger$  and  $F_2$ , respectively. Thus, a pair of these is redundant and can be left out, say  $F_2$  and its conjugate.

### 3.3 SUSY QM and constants of motion

In polar coordinates, the constants of motion for the TDHA given in [13] can be written as

$$G_\pm = \frac{1}{\sqrt{2|\tilde{E}|}} \rho e^{\pm i\varphi} \left\{ \left( \frac{1}{2} \pm L_z \right) \left( \frac{1}{\rho} \frac{\partial}{\partial \rho} \mp \frac{L_z}{\rho^2} \right) + \frac{1}{\rho} \right\}, \quad (62)$$

$$G_3 = L_z = -i \frac{\partial}{\partial \varphi} \quad (63)$$

and satisfy

$$[G_+, G_-] = 2G_3, \quad (64)$$

$$[G_\pm, G_3] = \mp G_\pm. \quad (65)$$

Thus,  $G_\pm$  and  $L_z$  define the  $su(2)$  Lie algebra. If we consider that the operators  $L_z$  and  $H_{h.a.}$  have simultaneous eigenfunctions, from equation (65) we show that

$$G_\pm \psi_{nm} \propto \psi_{n, m \pm 1}. \quad (66)$$

In order to find how the constants of motion of the TDHA and SUSY operators are related, we apply the operators  $G_\pm$  on the states given in (44). Therefore, we obtain

$$G_\pm \psi_{nm} = -\frac{1}{\sqrt{2|\tilde{E}|}} \left( m \pm \frac{1}{2} \right) \rho^{-\frac{1}{2}} e^{i(m \pm 1)\varphi} g_\pm^m u_{nm}, \quad (67)$$

where we have defined the radial operators

$$g_\pm^m = \mp \frac{d}{d\rho} + \frac{m \pm \frac{1}{2}}{\rho} - \frac{1}{m \pm \frac{1}{2}}. \quad (68)$$

Equations (56), (57) and (68) lead us to

$$F_1 = \frac{1}{\sqrt{2}} g_-^{m+1}, \quad (69)$$

$$F_1^\dagger = \frac{1}{\sqrt{2}} g_+^m. \quad (70)$$

Therefore, unlike to the TDIHO, the SUSY operators are directly contained in the spherical constants of motion.

## 4 The 2D potential $A\rho^{2\zeta-2} - B\rho^{\zeta-2}$ : zero-energy eigensubspace

The purpose of this section is to find the compact and non-compact algebras of a general potential which includes those of the TDIHO and TDHA as particular cases. We consider the following two-dimensional Hamiltonian [32, 33]

$$\mathcal{H} \equiv -\frac{1}{2} \left( \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2} \right) + A\rho^{2\zeta-2} - B\rho^{\zeta-2}, \quad (71)$$

where  $A, B > 0$ , and  $\zeta$  is a positive rational number. It has been shown that the zero-energy level for this operator possesses accidental degeneracy [32, 33, 34]. Let  $\Psi_0(\rho, \varphi)$  one state of the zero-energy eigensubspace, this is

$$\mathcal{H}\Psi_0 = 0. \quad (72)$$

If we define the operator

$$H \equiv \rho^{2-\zeta} \left[ -\frac{1}{2} \left( \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2} \right) + A\rho^{2\zeta-2} \right], \quad (73)$$

then from Eq.(72), it follows that

$$H\Psi_0(\rho, \varphi) = B\Psi_0(\rho, \varphi). \quad (74)$$

We note that Eq. (74) encompasses the Schrödinger equation for the 2D isotropic harmonic oscillator ( $\zeta = 2$ ) and the 2D hydrogen atom ( $\zeta = 1$ ).

By setting  $\rho^\zeta = \frac{\zeta}{2\sqrt{2A}} y^2$ , equation (74) is transformed to

$$\tilde{H}\Psi_0(y, \theta) \equiv \left[ -\frac{1}{2} \left( \frac{\partial^2}{\partial y^2} + \frac{1}{y} \frac{\partial}{\partial y} + \frac{1}{y^2} \frac{\partial^2}{\partial \theta^2} \right) + \frac{y^2}{2} \right] \Psi_0(y, \theta) = \Lambda \Psi_0(y, \theta), \quad (75)$$

where  $\theta = \frac{\zeta}{2}\varphi$ ,  $\Lambda = \frac{2B}{\zeta\sqrt{2A}}$  and  $\Psi_0(y, \theta) = R(y)e^{im\theta}$ . Eq. (75) corresponds to the Schrödinger equation of the TDIHO in polar coordinates  $(y, \theta)$ . This result will allow us to find the generators of the compact and non-compact symmetries of the operator  $H$ .

## 4.1 Constants of motion and spectrum generating algebras

Considering the last result, the constants of motion of the Hamiltonian  $\tilde{H}$  are obtained from Eqs. (31) and (32), and are given by

$$F_{\pm} = \mp i e^{\pm 2i\theta} \left[ (1 \pm L_{z'}) \left( \frac{1}{y} \frac{\partial}{\partial y} \mp \frac{L_{z'}}{y^2} \right) + \tilde{H} \right], \quad (76)$$

$$F_3 = -\frac{i}{2} \frac{\partial}{\partial \theta} \equiv \frac{L_{z'}}{2}, \quad (77)$$

whereas, from Eqs. (16) and (18), the Schrödinger operators are

$$K_{\pm} = \frac{1}{2} \left( \mp y \frac{d}{dy} + y^2 - 2K_3 \mp \frac{1}{2} \right), \quad (78)$$

$$K_3 = \frac{\tilde{H}}{2}. \quad (79)$$

By performing the change of variables  $(y, \theta) \rightarrow (\rho, \varphi)$ , the constants of motion are

$$\Theta_{\pm} = \mp \frac{i}{\zeta \sqrt{2A}} \rho^{2-\zeta} e^{\pm i\zeta\varphi} \left\{ \left( \frac{\zeta}{2} \pm L_z \right) \left( \frac{1}{\rho} \frac{\partial}{\partial \rho} \mp \frac{L_z}{\rho^2} \right) + \rho^{\zeta-2} H \right\}, \quad (80)$$

$$\Theta_3 = \frac{1}{\zeta} L_z \quad (81)$$

and the Schrödinger operators are

$$\Delta_3 = \frac{1}{2\zeta\sqrt{2A}} H, \quad (82)$$

$$\Delta_{\pm} = \frac{1}{2} \left( \mp \frac{2}{\zeta} \rho \frac{\partial}{\partial \rho} + \frac{2\sqrt{2A}}{\zeta} \rho^{\zeta} - 2\Delta_3 \mp 1 \right). \quad (83)$$

Notice that if  $R(\rho) \equiv \rho^{-\frac{1}{2}} U(\rho)$  then, for  $\zeta = 1, 2$  the operators above are reduced to the compact and the non-compact algebra generators for the 2D hydrogen atom and the 2D harmonic oscillator, respectively. On the other hand, by considering that  $\Theta_{\pm} = \Theta_1 \pm i\Theta_2$ , we obtain the operators which are essentially the symmetry generators reported in Ref. [34].

We show that the operators  $\Theta_{\pm}$  and  $\Theta_3$  satisfy the  $su(2)$  Lie algebra

$$[\Theta_+, \Theta_-] = 2\Theta_3, \quad (84)$$

$$[\Theta_{\pm}, \Theta_3] = \mp\Theta_{\pm}. \quad (85)$$

and the operators  $\Delta_{\pm}$  and  $\Delta_3$  close the  $so(2, 1)$  spectrum generating algebra

$$[\Delta_+, \Delta_-] = -2\Delta_3, \quad (86)$$

$$[\Delta_{\pm}, \Delta_3] = \mp\Delta_{\pm}. \quad (87)$$

By transforming the operator  $H$  to the extensively studied Hamiltonian of the TDIHO, we were capable to find its spherical constants of motion and the generators of the non-compact algebra.

## 5 Concluding remarks

The factorization techniques have been used to obtain algebraically the wave functions and the energy spectrum of many systems. In this paper, we showed in a systematic and clear way how factorization methods and hidden and dynamical symmetries are related. By means of the Schrödinger factorization we constructed the operators which close the  $so(2, 1)$  spectrum generating algebra for the TDIHO and the TDHA. We showed the link between the SUSY operators and the symmetry generators of the  $su(2)$  algebra for these systems.

The results concerning to the zero-energy eigensubspace for the  $2D$  potential  $A\rho^{2\zeta-2} - B\rho^{\zeta-2}$  were obtained by transforming (74) to the equation of the TDIHO. Thus, we found the origin of the hidden and dynamical symmetry generators and showed that these operators close the  $su(2)$  and  $so(2, 1)$  Lie algebra, respectively. We note that our non-compact generators and those given in [26] are different realizations of the algebra  $so(2, 1)$ . Also, our procedure clarifies the restrictions on the quantum numbers, the eigenfunctions and the origin of the symmetries generators reported in [26].

Although the compact and non-compact algebra generators are difficult to find, in this work we have shown that factorization methods provide the explicit form of these operators. This seems to be a first approach to a systematic method to find hidden and dynamical symmetries. Thus, we think that the determination of closed Lie algebras of a Hamiltonian is closer to science than art, contrary to the opinion expressed in [17].

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